Mise en température d'une tige cylindrique

① $\theta(x,0) = T_0 - T_1$ CI

CL : $\theta(0,t) = 0 = \theta(h,t)$

② Equation de la chaleur $D\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial T}{\partial t}$

Donc $\dfrac{\partial^2 \theta}{\partial x^2} D = \dfrac{\partial \theta}{\partial t}$

$\dfrac{\partial^2 \theta}{\partial x^2} = f(t)\, g''(x) \times D = f'(t)\, g(x)$

$\dfrac{g''(x)}{g(x)} = \dfrac{1}{D}\dfrac{f'(t)}{f(t)} \Rightarrow G(x) = F(t) \quad \forall x, t$

Donc $\dfrac{g''(x)}{g(x)} = K \Rightarrow g''(x) = K g(x)$

$g''(x) - K g(x) = 0$

Etude du signe de K.

Si $K > 0$, $g(x) = Ae^{-\sqrt{K}x} + Be^{+\sqrt{K}x}$

$\forall t > 0$, $\theta(x=0,t) = \theta(x=h,t) = 0$

Donc $g(0) = 0 \Rightarrow A + B = 0$

$g(h) = 0 \Rightarrow Ae^{-\sqrt{K}h} + Be^{+\sqrt{K}h} = 0$

$e^{-\sqrt{K}h} - e^{+\sqrt{K}h} = 0$

$-\sqrt{K}h = \sqrt{K}h \Rightarrow K = 0$

Impossible

Si $K < 0$, on pose $K = -k^2$ $k \in \mathbb{R}$

$g''(x) + k^2 g(x) = 0$

Solution de la forme

$g(x) = A\cos(kx) + B\sin(kx)$

$g(0) = 0 = A$

$g(h) = 0 = B\sin(kh) \Rightarrow kh = n\pi$ $n \in \mathbb{N}$

$g(x) = B\sin(k_n x)$ avec $\boxed{k_n = \dfrac{n\pi}{h}}$

$\dfrac{f'(t)}{f(t)} = -Dk_n^2$ $\quad f(t) = Ce^{-\frac{t}{\tau_n}}$

avec $\tau_n = \dfrac{1}{Dk_n^2}$

$\tau_n = \dfrac{h^2}{n^2\pi^2 D}$

Solution générale = superposition des modes propres :

$$\boxed{\theta(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\frac{t}{\tau_n}} \sin(k_n x)}$$

③ CI : $\theta(x, t=0) = T_0 - T_1 = \underbrace{\sum_{n=1}^{+\infty} b_n \sin(k_n x)}_{= F(x)}$

$F(x)$ est impaire.

④ $b_n = \dfrac{4\,\theta(x,0)}{n\pi} = \dfrac{4\,(T_0 - T_1)}{n\pi}$

$$\frac{\theta_n}{\theta_1} = \frac{4(T_0 - T_1)}{n\pi} e^{-\frac{t}{\tau_n}} \times \frac{\pi}{4(T_0 - T_1)} e^{\frac{t}{\tau_1}}$$

$$\boxed{\frac{\theta_n}{\theta_1} = \frac{1}{n} e^{-t\left(\frac{1}{\tau_n} - \frac{1}{\tau_1}\right)} \frac{\sin(k_n x)}{\sin(k_1 x)}}$$

$\dfrac{\theta_n}{\theta_1} \ll 1$ pour $t\left(\dfrac{1}{\tau_n} - \dfrac{1}{\tau_1}\right) \gg 1$

$$t \gg \frac{1}{\frac{1}{\tau_n} - \frac{1}{\tau_1}} = \frac{1}{\frac{\pi^2 D}{h^2}(n^2 - 1)}$$

le terme $n=3$ est négligeable devant le terme $n=1$ pour

$$t \gg \frac{h^2}{8\pi^2 D} = \frac{h^2 \rho c}{8\pi^2 \lambda}$$

$$t \gg \frac{1^2 \times 9000 \times \cancel{400}}{8 \times \pi^2 \times \cancel{400}} = 113\ s$$

On peut choisir $t_1 \sim 10^3\ s$